Int. J. Solids Structures Vol. 22, No. 11, pp. 1177–1187, 1986
Printed in Great Britain.

0020–7683/86 $3.00+.00
Pergamon Journals Ltd.

A HIGHER-ORDER BOUNDARY PERTURBATION METHOD FOR ASYMMETRIC DYNAMIC PROBLEMS IN SOLIDS—I. GENERAL FORMULATION†

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(*Received* 18 *February* 1985)

Abstract—A general formulation of a higher-order boundary perturbation method is developed for investigating two classes of asymmetry effects in solid bodies: eccentricity and ellipticity. General expressions for treating these problems are derived in terms of perturbed functions and perturbation parameters. In Part II, the method is applied to dynamic problems of radiation dislocation.

1. INTRODUCTION

The response of asymmetric systems often shows a strong departure from that of its symmetric counterparts especially if dynamic behaviour is concerned. Indeed, asymmetry, in general, is typical of a real physical situation and by ignoring it one may render an analysis inapplicable. However, asymmetric problems, by their very nature, lead to a more complex mathematical treatment than their axisymmetric analogues. Consequently, such problems have been treated, in recent years, by computer-oriented methods of which finite elements (see, e.g. Datta *et al.*[1]) or the extended boundary condition method (see, e.g. Ott and Fang[2]) are typically representative. These methods are not usually intended to provide a generalized description of the responses or to indicate their analytical structure. In this paper, we develop a higher-order boundary perturbation method (BPM) which represents an alternative approach to an exact analytic treatment or to numerical approaches.

A systematic development of the boundary perturbation method is given in Morse and Feshbach[3]. The method has been used extensively in applications to fluid mechanics; treatments are given by Van Dyke[4] and applications to the study of water waves have been presented by Stoker[5]. In recent years, the BPM also has been used to treat problems of surface perturbations in reactor physics (see, e.g. Komata[6]). In the field of mechanics of solids, however, it appears that the BPM has had but limited use. A review of works originally published in Russian has been given by Kubenko *et al.*[7].

In the present investigation we turn to the solution of some asymmetric problems which appear in solid mechanics. We present, in Part I of this paper, a general formulation of the basic equations which will permit the application of the BPM to a variety of problems which are subjected to Dirichlet or Neumann boundary conditions. Two classes of asymmetric problems are treated: those concerned with the effects of eccentricity and of ellipticity.

In the first class, we treat bodies defined in a circular domain containing forcing (or source) terms which are eccentric to the centre. Classically, exact solutions to this type of problem often result in a mathematical treatment requiring bipolar coordinate systems. In the second class, we consider elliptic domains. Problems associated with the mechanics of such bodies often necessitate a resort to Mathieu functions, and in particular, for dynamic problems of this kind, exact solutions do not lead readily to an identification of resonant

† Part of this work was performed by the first author at the Laboratoire de Mécanique des Solides, Ecole Polytechnique, Palaiseau, France, with the support of CNRS.

behaviour. The use of the BPM, however, yields relatively simple solutions for which resonant frequencies are easily obtained.

We first outline, in Section 2, the general development of the BPM, applicable to the two indicated classes of problems. Relations governing the geometry are established and explicit expressions, applicable up to a third-order BPM for the class of eccentric problems, are derived in Section 3. We develop, in Section 4, explicit expressions for the somewhat more complex problem of the ellipse which are applicable to a higher-order scheme. To the best knowledge of the authors, such expressions do not appear in the literature.

In Part II of this paper[8] the BPM is applied to two dynamic problems. We consider first the radiation due to a screw dislocation oscillating about an eccentric point of a circular cylinder. The BPM solution is characteristically simple and yet for moderate eccentricities, leads to results which are in good agreement with the exact solution recently given by the authors[9]. In the second problem investigated, we consider the radiation due to a screw dislocation oscillating about the centre of an elliptic cylinder.

2. GENERAL DEVELOPMENT OF THE BPM

Before proceeding with explicit expressions and details, we first outline the general ideas which will be used below in developing the perturbation method for the two classes of problems under discussion.

We consider first the development for the eccentric problem. The given boundary value problem for a Green's function consists here of a governing differential equation

$$L[f(\bar{r},\bar{\theta})] = P\delta(\bar{r}-l)\delta(\bar{\theta}) \tag{2.1a}$$

valid within a domain defined by a circle C_0 of radius a (centred at $\bar{O}$) on which a set of appropriate boundary conditions of a general form

$$\bar{B}_i|_{C_0} = \bar{B}_i[f(a,\bar{\theta})] = 0, \qquad i = 1,2,\ldots, \tag{2.1b}$$

are prescribed (Fig. 1). In the above, L is a linear differential operator, the set $\bar{B}_i$ contains the function f and its derivatives at $\bar{r} = a$, and δ is the Dirac-delta function. Defining the eccentricity parameter by

$$\eta = l/a, \tag{2.2}$$

we note that due to an eccentricity $\eta > 0$ in the particular term of eqn (2.1a), the solution is $\bar{\theta}$-dependent irrespective of a $\bar{\theta}$-independent $\bar{B}_i$. Classical solutions consist of expanding eqns (2.1) as Fourier series in $\bar{\theta}$ and solving a *set of inhomogeneous equations* with homogeneous boundary conditions.

To proceed with the boundary perturbation method we establish an eccentric polar coordinate system (r, θ) centred at 0, a distance $l = \eta a$, and construct a fictitious circle C_a

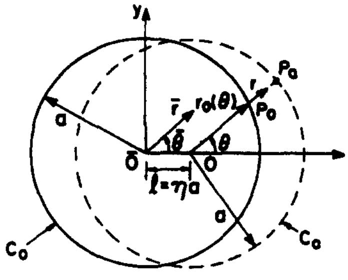

Fig. 1.

of radius $r = a$ about this point. If η is not too large, the original curve C_0 can then be considered as a perturbation of the curve C_a, with points P_0 on C_0 being perturbed points of P_a on C_a (Fig. 1). We note that C_0 is then a curve with varying radial distance from 0, i.e. $r_0 = r_0(\theta)$. Symbolically we may write the perturbed relation, $C_a \to C_0$, as

$$a \to r_0 = r_0(a, \theta, \eta), \tag{2.3}$$

with $r_0|_{\eta=0} = a$.

We assume now that the function f is equally valid and analytic within the domain defined by C_a. The boundary value problem then becomes

$$L[f(r,\theta)] = P\delta(r) \tag{2.4a}$$

with boundary conditions

$$B_i|_{C_a} = B_i[f(r,\theta)]_{C_a}, \tag{2.4b}$$

where $B_i|_{C_a}$ must yet be related to $\bar{B}_i|_{C_0}$ as discussed below.

Following generally developed methods of perturbation theory we let

$$f(r,\theta) = f^{(0)} + \eta f^{(1)} + \eta^2 f^{(2)} + \cdots = \sum_{j=0}^{N} \eta^j f^{(j)}(r,\theta) \tag{2.5}$$

for an Nth order scheme.

Using the linearity property, eqn (2.4a) is written as

$$L[f^{(0)}] + \eta L[f^{(1)}] + \eta^2 L[f^{(2)}] + \cdots = P\delta(r) \tag{2.6}$$

which is valid for $\eta > 0$. Equation (2.6) is satisfied by setting

$$L[f^{(0)}] = P\delta(r), \tag{2.7a}$$

$$L[f^{(j)}] = 0, \qquad j = 1, 2, \ldots, N. \tag{2.7b}$$

We now turn our attention to the boundary condition eqn (2.4b). Noting that the points P_0 and P_a possess the same coordinate θ, the original boundary conditions $\bar{B}_i|_{C_0}$ at P_0 can be expressed in terms of combinations of $B_i|_{C_a}$ on the fictitious curve by means of a Taylor series expansion in powers of $(a - r_0)$ which in turn are expressed as powers of η for any fixed θ and radius a. Thus, we can expand $\bar{B}_i|_{C_0}$ as a series

$$\bar{B}_i|_{C_0} = B_i^{(0)}|_{C_a} + \eta B_i^{(1)}|_{C_a} + \eta^2 B_i^{(2)}|_{C_a} + \cdots + \eta^N B_i^{(N)}|_{C_a}, \tag{2.8a}$$

where

$$B_i^{(j)}|_{C_a} = B_i^{(j)}[f^{(k)}(\theta, a)], \qquad j = 1, 2, \ldots, N; \quad k = 1, 2, \ldots, (j-1) \tag{2.8b}$$

are θ-dependent coefficients.

Now, from eqn (2.1b) we note that $\bar{B}_i|_{C_0} = 0$. Since this is true for all η, we require that

$$B_i^{(j)}|_{C_a} = 0, \qquad j = 0, 1, 2, \ldots, N. \tag{2.9}$$

Equation (2.9) represents the appropriate boundary conditions for the corresponding differential equations, eqns (2.7).

Thus, to summarize, the boundary perturbation method for eccentric problems as

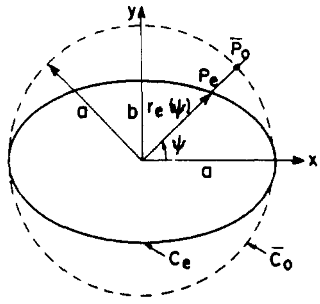

Fig. 2.

presented here is reduced to satisfying a single inhomogeneous differential equation†

$$L[f^{(0)}] = P\delta(r) \qquad \text{with} \quad B_i^{(0)} = B_i^{(0)}|_{C_a}; \qquad j = 0, \tag{2.10a}$$

and then solving sequentially a set of simple boundary value problems

$$L[f^{(j)}] = 0, \qquad B_i^{(j)} = B_i^{(j)}|_{C_a}; \qquad j = 1, 2, \ldots, N, \tag{2.10b}$$

consisting of homogeneous differential equations.

The $j = 0$ problem is, under usual circumstances, an axisymmetric problem, often with a known solution. Consequently, the effort required is to solve, in sequence, the $j \geqslant 1$ problems which according to eqn (2.8b) possess boundary conditions $B_i^{(j)}$ evaluated from the previously obtained solutions $f^{(k)}$, $k = 0, 1, \ldots, (j-1)$. The perturbation method, as will be seen below, usually leads to relatively simple boundary conditions.

We now turn to the problem of an elliptic domain for which the general ideas are basically similar. However, for this case it is not necessary to establish two separate coordinate systems: a single (r, ψ) coordinate is defined, with origin at the centre of the ellipse whose semimajor and -minor axes, are a and b, respectively (Fig. 2). The ellipticity ε is defined as

$$\varepsilon = a/b - 1, \qquad \varepsilon > 0. \tag{2.11}$$

The basic Green's function problem then consists of the differential equation

$$L[f(r, \psi)] = P\delta(r) \tag{2.12a}$$

valid within a domain defined by C_ε, the curve of the ellipse, on which appropriate boundary conditions, of a general form

$$\bar{B}_i|_{C_\varepsilon} = 0, \qquad i = 1, 2, \ldots, <m, \tag{2.12b}$$

are prescribed.

Proceeding now with the BPM for this case, we consider, for a moderate value ε, the curve C_ε to be a perturbation of a circumscribing circle $\bar{C}_0$ of radius a, with points P_ε on C_ε being perturbed points of $\bar{P}_0$ on $\bar{C}_0$ (Fig. 2). The ellipse C_ε is then, as in the previous case, a curve with varying radial distance from 0, i.e. $r_\varepsilon = r_\varepsilon(\psi)$. Symbolically, the perturbed relation $\bar{C}_0 \rightarrow C_\varepsilon$ may be written as

$$a \rightarrow r_\varepsilon = r_\varepsilon(a, \psi, \varepsilon), \tag{2.13}$$

with $r_\varepsilon|_{\varepsilon=0} = a$.

† Alternative cases may exist where, instead of the particular term of eqn (2.10a), the "forcing term" of the system is replaced by a specified singularity within the body (see, e.g. Part II [8]).

The method proceeds exactly as in the previous case where in the above [eqns (2.4) to (2.10)], C_a is replaced everywhere by $\bar{C}_0$, and the perturbation parameter η is replaced by ε.

For the present case of the ellipse, the $j = 0$ problem is, under usual circumstances, an axisymmetric problem for the circular domain, with a known solution. The sequence of problems, $j \geq 1$ which must be solved, again possesses boundary conditions $B_i^{(j)}$ which have been evaluated from previously obtained k solutions, $k = 1, 2, \ldots, (j-1)$.

In the following sections we derive the explicit expressions required to apply the BPM to eccentric and elliptic problems.

3. GEOMETRIC RELATIONS AND PERTURBATION EXPRESSIONS FOR ECCENTRIC PROBLEM

3.1. *Geometric relations*

We consider a circle C_0 of radius a, centred about point $\bar{O}$. A polar coordinate system (r, θ) is established with origin O located at a distance l to the right of $\bar{O}$ (Fig. 3). Defining $r_0 = r_0(\theta)$ as the variable radial distance to C_0, we note that

$$r_0^2 + l^2 + 2r_0 l \cos\theta = a^2 \tag{3.1}$$

from which

$$r_0(\theta) = -l\cos\theta + (a^2 - l^2\sin^2\theta)^{1/2}. \tag{3.2a}$$

Recalling that $l = \eta a$,

$$r_0/a = -\eta\cos\theta + (1 - \eta^2\sin^2\theta)^{1/2}. \tag{3.2b}$$

We denote the normal and tangential directions to C_0 by n and s, where n is inclined with respect to the x-axis at an angle α (Fig. 3).

Consider now a function $f = f(r, \theta)$ defined in the (r, θ) polar coordinate system. Directional derivatives in the n and s direction are then given by

$$\left.\frac{\partial f}{\partial n}\right|_{C_0} = \left.\frac{\partial f}{\partial r}\right|_{C_0} \cos(\theta - \alpha) - \frac{1}{r_0}\left.\frac{\partial f}{\partial \theta}\right|_{C_0} \sin(\theta - \alpha), \tag{3.3a}$$

$$\left.\frac{\partial f}{\partial s}\right|_{C_0} = \left.\frac{\partial f}{\partial r}\right|_{C_0} \sin(\theta - \alpha) + \frac{1}{r_0}\left.\frac{\partial f}{\partial \theta}\right|_{C_0} \cos(\theta - \alpha). \tag{3.3b}$$

Applying the sin law to the triangle $\bar{O}P_0O$, we obtain

$$\sin(\theta - \alpha) = \eta\sin\theta \tag{3.4}$$

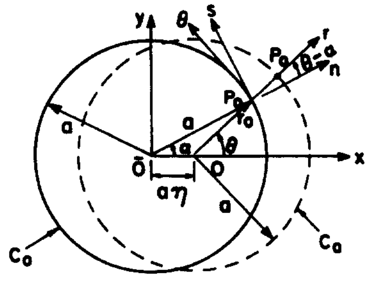

Fig. 3.

from which

$$\cos(\theta-\alpha) = (1-\eta^2 \sin^2\theta)^{1/2}. \tag{3.5}$$

3.2. *Approximations for eccentricity* $\eta < 1$

We note first that the sine term is exactly a linear function of η. In our treatment below, consistent with a third-order scheme, we expand up to powers η^3. Expanding thus eqn (3.5), we obtain

$$\cos(\theta-\alpha) = 1 - \frac{\sin^2\theta}{2}\eta^2 + O(\eta^4). \tag{3.6}$$

Performing a similar expansion on eqn (3.2b) yields

$$r_0/a = 1-\cos\theta\eta - \frac{\sin^2\theta}{2}\eta^2 + O(\eta^4) \tag{3.7}$$

from which

$$1/r_0 = \frac{1}{a}\{1+\cos\theta\eta + 1/2(1+\cos^2\theta)\eta^2 + \cos\theta\eta^3\} + O(\eta^4). \tag{3.8}$$

Substitution of eqns (3.4), (3.6) and (3.8) in eqns (3.3) then leads to

$$\left.\frac{\partial f}{\partial \eta}\right|_{C_0} = f_{,r}|_{C_0} - \frac{\sin\theta}{a} f_{,\theta}|_{C_0}\eta - \sin\theta\left(\sin\theta f_{,r} + \frac{\cos\theta}{a} f_{,\theta}\right)\Big|_{C_0}\eta^2 - \frac{\sin\theta}{2a}(1+\cos^2\theta) f_{,\theta}|_{C_0}\eta^3 + O(\eta^4) \quad (3.9)\dagger$$

and

$$\left.\frac{\partial f}{\partial s}\right|_{C_0} = \left.\frac{f_{,\theta}}{a}\right|_{C_0} + \left(\sin\theta f_{,\theta} + \frac{\cos\theta}{a} f_{,r}\right)\Big|_{C_0}\eta + \frac{\cos^2\theta}{a} f_{,\theta}|_{C_0}\eta^2 + \frac{\cos\theta}{2a}(1+\cos^2\theta) f_{,\theta}|_{C_0}\eta^3 + O(\eta^4). \tag{3.10}$$

3.3. *Perturbed functions and directional derivatives*

In the boundary perturbation scheme as discussed above, we obtain solutions to problems posed in the domain defined by C_0 as perturbations from the circle C_a of radius a centred about point O. More precisely, we consider functions at a point P_0 ($r = r_0, \theta$) as perturbed values of the function existing at P_a ($r = a, \theta$) (Fig. 3). Noting that both P_0 and P_a possess the same coordinate θ, we may expand any function $g(r, \theta)$ about the point P_a by means of a Taylor series

$$g(r_0,\theta) = g(a,\theta) + g_{,r}(a)\cdot(r_0-a) + \frac{g_{,rr}(a,\theta)}{2}(r_0-a)^2 + \frac{g_{,rrr}(a,\theta)}{6}(r_0-a)^3 + \cdots. \tag{3.11}$$

Substituting eqn (3.7) in eqn (3.11) and collecting like terms in η leads to

$$g(r_0,\theta) = g_a - a\cos\theta g_{a,r}\eta + \frac{a}{2}(-\sin^2\theta g_{a,r} + a\cos^2\theta g_{a,rr})\eta^2 + \frac{a^2\cos\theta}{6}(3\sin^2\theta g_{a,rr} - a\cos^2\theta g_{a,rrr})\eta^3 + O(\eta^4). \tag{3.12}$$

† Here, and in all subsequent expressions, derivatives with respect to a variable are denoted by a subscript preceded by a comma, e.g. $f_{,r} \equiv \partial f/\partial r$, etc.

In the above $g_a \equiv g(a,\theta)$, $g_{a,r} \equiv g_{,r}(r=a,\theta)$, etc. We note too that the function g can represent a function $f(r,\theta)$, $f_{,r}(r,\theta)$, $f_{,\theta}(r,\theta)$ and higher derivatives.

In anticipation of the boundary perturbation scheme (third-order), we now postulate a desired function to be of the form

$$f(r,\theta) = f^{(0)}+\eta f^{(1)}+\eta^2 f^{(2)}+\eta^3 f^{(3)}, \tag{3.13}$$

where $f^{(j)} = f^{(j)}(r,\theta)$.

Substituting the function $f(r,\theta)$ and its derivatives as given by eqns (3.13) in (3.12), $f(r,\theta)$ on C_0 is expressed in terms of evaluated quantities on C_a as follows:

$$f|_{C_0} = f^{(0)}+[f^{(1)}+{}_0\Phi_1^{(0)}]\eta+[f^{(2)}+{}_0\Phi_1^{(1)}+{}_0\Phi_2^{(0)}]\eta^2+[f^{(3)}+{}_0\Phi_1^{(2)}+{}_0\Phi_2^{(1)}+{}_0\Phi_3^{(0)}]\eta^3, \tag{3.14}$$

where ${}_0\Phi_k^{(j)}$ are functions of $f^{(j)}$ and its derivatives as well as of the coordinate θ. Explicit expressions for the ${}_0\Phi_k^{(j)}$ terms appearing above are given in the Appendix, eqns (A.1). We note that these terms are evaluated on the curve C_a, i.e. at $r = a$.

Performing a similar operation and substituting in eqns (3.9) and (3.10) we obtain, after considerable manipulation, the following expression for derivatives normal and tangential to the curve C_0:

$$\left.\frac{\partial f}{\partial n}\right|_{C_0} = f_{,r}^{(0)}+[f_{,r}^{(1)}+{}_n\Phi_1^{(0)}]\eta+[f_{,r}^{(2)}+{}_n\Phi_1^{(1)}+{}_n\Phi_2^{(0)}]\eta^2+[f_{,r}^{(3)}+{}_n\Phi_1^{(2)}+{}_n\Phi_2^{(1)}+{}_n\Phi_3^{(0)}]\eta^3, \tag{3.15}$$

$$\left.\frac{\partial f}{\partial s}\right|_{C_0} = \frac{1}{a}f_{,\theta}^{(0)}+\left[\frac{1}{a}f_{,\theta}^{(1)}+{}_s\Phi_1^{(0)}\right]\eta+\left[\frac{1}{a}f_{,\theta}^{(2)}+{}_s\Phi_1^{(1)}+{}_s\Phi_2^{(0)}\right]\eta^2$$
$$+\left[\frac{1}{a}f_{,\theta}^{(3)}+{}_s\Phi_1^{(2)}+{}_s\Phi_2^{(1)}+{}_s\Phi_3^{(0)}\right]\eta^3. \tag{3.16}$$

The expressions for ${}_n\Phi_k^{(j)}$ and ${}_s\Phi_k^{(j)}$ $(k, j = 1,2,3)$ are given by eqns (A.2) and (A.3), respectively, in the Appendix.

We note that the above equations, eqns (3.14)–(3.16), express a function $f(r,\theta)$ and its directional derivatives in the normal and tangential direction to C_0 at a point P_0 in terms of the usual radial and circumferential derivatives of $f(r,\theta)$ existing at the corresponding point P_a on C_a.

Having established the above relations, we may readily apply the BPM to any problem of this class whose boundary conditions are specified by a function and/or normal and tangential derivatives on the boundary. An application of the BPM to the dynamic problem of a dislocation which oscillates about an eccentric point of a circular elastic rod is given in Part II [8].

4. GEOMETRIC RELATIONS AND PERTURBATION EXPRESSIONS FOR ELLIPTIC PROBLEMS

4.1. *Geometric relations and expansions*

We consider an ellipse C_e located in the x–y plane defined by the equation (Fig. 2)

$$x^2/a^2+y^2/b^2 = 1, \qquad a > b. \tag{4.1}$$

Denoting the ellipticity ε as

$$\varepsilon = a/b-1, \qquad \varepsilon > 0, \tag{4.2}$$

eqn (4.1) is rewritten as

$$x^2+\gamma^2 y^2 = a^2, \tag{4.3}$$

where

$$\gamma = 1+\varepsilon. \tag{4.4}$$

Solutions are obtained by considering the ellipse as the perturbed boundary with respect to the circumscribing circle, $\bar{C}_0$, defined by

$$x_0^2+y_0^2 = a^2 \tag{4.5a}$$

or

$$x_0 = a\cos\psi, \qquad y_0 = a\sin\psi, \tag{4.5b, c}$$

where x_0, y_0 represent the coordinates of a generic point $\bar{P}_0$ on $\bar{C}_0$.

Generic points P_ε on the ellipse C_ε may similarly be defined by the coordinates

$$x = r_\varepsilon(\psi)\cos\psi, \quad y = r_\varepsilon(\psi)\sin\psi, \tag{4.6a, b}$$

where r_ε represents the variable radial distance from the origin to P_ε. Substituting eqn (4.6) in (4.3) and using eqn (4.4) we obtain the *exact* expression

$$r_\varepsilon^2 = a^2[1+(2\varepsilon+\varepsilon^2)\sin^2\psi]^{-1}. \tag{4.7}$$

Considering the case of moderately small ellipticity ε, consistent with a second-order method, we perform expansions in ε, here and below, up to order ε^2.

Thus, eqn (4.7) yields

$$r_\varepsilon(\psi) = a\left[1-\varepsilon\sin^2\psi+\frac{\varepsilon^2}{2}\sin^2\psi(2\sin^2\psi-\cos^2\psi)\right]+O(\varepsilon^3). \tag{4.8}$$

We note, in passing, the simple relation

$$(r_\varepsilon-a)^2 = a^2\sin^4\psi\varepsilon^2+O(\varepsilon^3), \tag{4.9a}$$

which subsequently will prove useful as well as the relation

$$1/r_\varepsilon = \frac{1}{a}\left(1+\varepsilon\sin^2\psi+\frac{\varepsilon^2}{8}\sin^2 2\psi\right)+O(\varepsilon^3). \tag{4.9b}$$

At any point P_ε on the ellipse C_ε, the normal n to the ellipse, defined by the angle α (Fig. 4) is given by

$$\tan\alpha = -(\mathrm{d}y/\mathrm{d}x)^{-1}, \tag{4.10a}$$

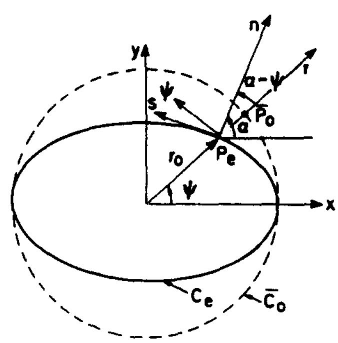

Fig. 4.

which, using eqn (4.2), becomes

$$\tan\alpha = \frac{a\gamma}{x}(1-x^2/a^2)^{1/2}. \tag{4.10b}$$

Substituting eqns (4.6a) and (4.8) in eqn (4.10b) and expanding, we obtain finally

$$\tan\alpha = (1+\varepsilon)^2 \tan\psi + O(\varepsilon^3). \tag{4.11}$$

Making use of standard trigonometric relations, and retaining terms up to $O(\varepsilon^2)$ in the expansion, it follows that

$$\sin\alpha = \sin\psi[1+2\cos^2\psi\varepsilon+\cos^2\psi(1-6\sin^2\psi)\varepsilon^2]+O(\varepsilon^3), \tag{4.12a}$$

$$\cos\alpha = \cos\psi[1-2\sin^2\psi\varepsilon-3\sin^2\psi\cos 2\psi\varepsilon^2]+O(\varepsilon^3). \tag{4.12b}$$

From these, we may immediately determine the following relations to be used below:

$$\sin(\alpha-\psi) = \sin\psi\cos\psi\,[2\varepsilon+(\cos^2\psi-3\sin^2\psi)\varepsilon^2]+O(\varepsilon^3), \tag{4.13a}$$

$$\cos(\alpha-\psi) = 1-\tfrac{1}{2}(\sin 2\psi)^2\varepsilon^2+O(\varepsilon^3). \tag{4.13b}$$

Consider now any analytic function $f = f(r,\psi)$ defined in the polar coordinate system. Directional derivatives in the directions n and s, normal and tangential to the ellipse C_ε at a point P_ε are then given by

$$\left.\frac{\partial f}{\partial n}\right|_{C_\varepsilon} = \left.\frac{\partial f}{\partial r}\right|_{C_\varepsilon}\cos(\alpha-\psi)+\frac{1}{r_\varepsilon}\left.\frac{\partial f}{\partial\psi}\right|_{C_\varepsilon}\sin(\alpha-\psi), \tag{4.14a}$$

$$\left.\frac{\partial f}{\partial s}\right|_{C_\varepsilon} = \frac{1}{r_\varepsilon}\left.\frac{\partial f}{\partial\psi}\right|_{C_\varepsilon}\cos(\alpha-\psi)-\left.\frac{\partial f}{\partial r}\right|_{C_\varepsilon}\sin(\alpha-\psi). \tag{4.14b}$$

Substituting eqns (4.9b) and (4.13), and retaining terms up to the required order, yields

$$\left.\frac{\partial f}{\partial n}\right|_{C_\varepsilon} = f_{,r}|_{C_\varepsilon}+\sin 2\psi\left.\left(\frac{1}{a}f_{,\psi}-\frac{1}{2}\sin 2\psi f_{,r}\right)\right|_{C_\varepsilon}\varepsilon+\frac{\sin 4\psi}{4a}f_{,\psi}|_{C_\varepsilon}\varepsilon^2+O(\varepsilon^3), \tag{4.15a}$$

$$\left.\frac{\partial f}{\partial s}\right|_{C_\varepsilon} = \frac{1}{a}f_{,\psi}|_{C_\varepsilon}+\left.\left(\frac{\sin^2\psi}{a}f_{,\psi}-\sin 2\psi\, f_{,r}\right)\right|_{C_\varepsilon}\varepsilon$$
$$+\frac{\sin 2\psi}{2}\left.\left[(3\sin^2\psi-\cos^2\psi)f_{,r}-\frac{3}{4a}\sin 2\psi\, f_{,\psi}\right]\right|_{C_\varepsilon}\varepsilon^2+O(\varepsilon^3). \tag{4.15b}$$

4.2. *Perturbed functions and their directional derivatives*

As discussed in Section 2, we consider points P_ε on the ellipse C_ε to be perturbations of points $\bar{P}_0$ on the circumscribing circle $\bar{C}_0$ (Fig. 4). Noting that here P_ε and $\bar{P}_0$ possess the same coordinate ψ, we may expand any function $g(r,\psi)$ about $\bar{P}_0$ by means of a Taylor series

$$g(r_\varepsilon,\psi) = g(a,\psi)+g_{,r}(a,\psi)\cdot(r_\varepsilon-a)+\frac{g_{,rr}(a,\psi)}{2}\cdot(r_\varepsilon-a)^2, \tag{4.16}$$

where again g represents either f or its derivatives $f_{,r}$, $f_{,\psi}$, etc.

Substituting eqn (4.16) in eqn (4.15) and using eqns (4.8) and (4.9a),

$$g(r_e,\psi) = g_a-(a\sin^2\psi g_{a,r})\varepsilon+\frac{a}{2}\sin^2\psi[(2\sin^2\psi-\cos^2\psi)g_{a,r}+a\sin^2\psi g_{a,rr}]\varepsilon^2+O(\varepsilon^3), \tag{4.17}$$

where $g_a \equiv g(a,\psi)$, etc.

In accordance with the second-order perturbation scheme we now consider a function $f(r,\psi)$ of the form

$$f(r,\psi) = f^{(0)}(r,\psi)+\varepsilon f^{(1)}(r,\psi)+\varepsilon^2 f^{(2)}(r,\psi). \tag{4.18}$$

Substituting in eqn (4.17) and collecting terms of the same order in ε, we obtain

$$f|_{C_e} = f^{(0)}|_{C_0}+\{f^{(1)}+{}_0\Psi_1^{(0)}\}\varepsilon+\{f^{(2)}+{}_0\Psi_1^{(1)}+{}_0\Psi_2^{(0)}\}\varepsilon^2, \tag{4.19}$$

where $f^{(j)}$ and ${}_0\Psi_k^{(j)}$, as given by eqn (A.4) in the Appendix, are evaluated on $\bar{C}_0$, i.e. at $r = a$.

Similarly, substituting eqn (4.18) in eqn (4.15), making use of eqn (4.17) and collecting all terms of same order in ε, yields

$$\left.\frac{\partial f}{\partial n}\right|_{C_e} = f_{,r}^{(0)}|_{C_0}+\{f_{,r}^{(1)}+{}_n\Psi_1^{(0)}\}\varepsilon+\{f_{,r}^{(2)}+{}_n\Psi_1^{(1)}+{}_n\Psi_2^{(0)}\}\varepsilon^2, \tag{4.20a}$$

$$\left.\frac{\partial f}{\partial s}\right|_{C_e} = \frac{1}{a}f_{,\psi}^{(0)}|_{C_0}+\left\{\frac{1}{a}f_{,\psi}^{(1)}+{}_s\Psi_1^{(0)}\right\}\varepsilon+\left\{\frac{1}{a}f_{,\psi}^{(2)}+{}_s\Psi_1^{(1)}+{}_s\Psi_2^{(0)}\right\}\varepsilon^2, \tag{4.20b}$$

where ${}_n\Psi_k^{(j)}$ and ${}_s\Psi_k^{(j)}$ $(j,k = 1,2)$ are given respectively, by eqns (A.5), (A.6) in the Appendix.

We note that all quantities appearing on the right-hand side of eqns (4.19) and (4.20) are evaluated at points $\bar{P}_0(a,\psi)$ on the circle $\bar{C}_0$. Thus, these equations express the value of a function at a perturbed point P_e on C_e in terms of the quantities at the point $\bar{P}_0$.

An application of the above equations to a dynamic problem in an elliptic domain is given in Part II [8] of this paper.

REFERENCES

1. S. K. Datta, K. C. Wong and A. H. Shah, Dynamic stresses and displacements around cylindrical cavities of arbitrary shape. *J. Appl. Mech.* **51** (4), 798–803 (Dec. 1984).
2. R. H. Ott and D. J. Fang, Comparison of the extended boundary condition method and perturbation method for scattering from a sinusoidal sea surface. *J. Math. Phys.* **24** (8), 2240–2249 (Aug. 1983).
3. P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, Vol. 2. McGraw-Hill, New York (1953).
4. M. Van Dyke, *Perturbation Methods in Fluid Mechanics*. Academic Press, New York (1964).
5. J. J. Stoker, *Water Waves*. Interscience, New York (1957).
6. M. Komata, A generalized perturbation theory applicable to reactor boundary changes. *Nucl. Sci. Engng* **64**, 811–822 (1977).
7. V. D. Kubenko, Yu. N. Nemish, K. I. Shnerneko and N. A. Shul'ga, Perturbation method in boundary value problems of the mechanics of deformable bodies. *Prikl. Mekh.* **18** (11), 3–20 (Nov. 1982). (English transl. in *Soviet Appl. Mech.* **18**, 955–970.)
8. R. Parnes and A. I. Beltzer, A higher-order boundary perturbation method for asymmetric dynamic problems in solids—II. Application to dislocation radiation. *Int. J. Solids Structures* **22**, 1189–1199 (1986).
9. A. I. Beltzer and R. Parnes, Resonance radiation from imperfections—I. A discrete dislocation. *Mech. Mater.* **3**, 199–210 (1984).

APPENDIX: BOUNDARY CONDITION COEFFICIENTS

(a) *Eccentric problems*

The boundary condition coefficients, ${}_0\Phi_k^{(j)}$, ${}_n\Phi_k^{(j)}$ and ${}_s\Phi_k^{(j)}$ appearing in eqns (3.14)–(3.16), and obtained by substituting eqn (3.13) in eqns (3.12), (3.9) and (3.10), respectively, are given as follows for $j, k = 1, 2, 3$:

$$\text{(a)}\quad {}_0\Phi_1^{(j)} = -a\cos\theta f_{,r}^{(j)},$$

$$\text{(b)}\quad {}_0\Phi_2^{(j)} = \frac{a}{2}[a\cos^2\theta f_{,rr} - \sin^2\theta f_{,r}]^{(j)},$$

$$\text{(c)}\quad {}_0\Phi_3^{(j)} = \frac{a^2}{2}\cos\theta\left[\sin^2\theta f_{,rr} - \frac{a}{3}\cos^2\theta f_{,rrr}\right]^{(j)}, \tag{A.1}$$

$$\text{(a)}\quad {}_n\Phi_1^{(j)} = -\left[a\cos\theta f_{,rr} + \frac{\sin\theta}{a} f_{,\theta}\right]^{(j)},$$

$$\text{(b)}\quad {}_n\Phi_2^{(j)} = \left[\frac{a^2}{2}\cos^2\theta f_{,rrr} - \frac{a\sin^2\theta}{2} f_{,rr} - \frac{\sin^2\theta}{2} f_{,r} + \frac{\sin 2\theta}{2} f_{,r\theta} - \frac{\sin 2\theta}{2a} f_{,\theta}\right]^{(j)},$$

$$\text{(c)}\quad {}_n\Phi_3^{(j)} = \left[-\frac{a^3}{6}\cos^3\theta f_{,rrrr} + \frac{a}{4}\sin\theta\sin 2\theta(af_{,rrr} + f_{,rr})\right.$$
$$\left. - \frac{a}{4}\sin 2\theta\cos\theta f_{,rr\theta} + \sin\theta\left(\frac{\sin^2\theta}{2} + \cos^2\theta\right) f_{,r\theta} - \frac{\sin\theta}{2a}(1+\cos^2\theta) f_{,\theta}\right]^{(j)}, \tag{A.2}$$

$$\text{(a)}\quad {}_s\Phi_1^{(j)} = \left[\frac{\cos\theta}{a} f_{,\theta} - \cos\theta f_{,\theta r} + \sin\theta f_{,r}\right]^{(j)},$$

$$\text{(b)}\quad {}_s\Phi_2^{(j)} = \left[\frac{a}{2}\cos^2\theta f_{,\theta rr} - \frac{a\sin 2\theta}{2} f_{,rr} - \left(1 - \frac{\sin^2\theta}{2}\right) f_{,\theta r} + \frac{\cos^2\theta}{a} f_{,\theta}\right]^{(j)},$$

$$\text{(c)}\quad {}_s\Phi_3^{(j)} = \left[\frac{a^2}{4}\cos\theta\sin 2\theta f_{,rrr} - \frac{a^2}{6}\cos^3\theta f_{,\theta rrr} - \frac{a\sin^3\theta}{2} f_{,rr}\right.$$
$$\left. + \frac{a}{2}\cos\theta f_{,\theta rr} - \frac{\cos\theta}{2} f_{,\theta r} + \frac{\cos\theta}{2a}(1+\cos^2\theta) f_{,\theta}\right]^{(j)}. \tag{A.3}$$

Bracketed terms, $[\ldots]^{(j)}$, appearing above, denote that the combination of functions and derivatives contained within refer to the function $f^{(j)}$.

(b) *Elliptic problems*

The boundary condition coefficients, ${}_0\Psi_k^{(j)}, {}_n\Psi_k^{(j)}$ and ${}_s\Psi_k^{(j)}$, $(j, k = 1,2)$ appearing in eqns (4.19) and (4.20), and obtained by substituting eqn (4.18) in eqns (4.17) and (4.15), respectively, are given as follows:

$$\text{(a)}\quad {}_0\Psi_1^{(j)} = -a\sin^2\psi f_{,r}^{(j)},$$

$$\text{(b)}\quad {}_0\Psi_2^{(j)} = \frac{a}{2}\sin^2\psi[(2\sin^2\psi - \cos^2\psi) f_{,r} + a\sin^2\psi f_{,rr}]^{(j)}, \tag{A.4}$$

$$\text{(a)}\quad {}_n\Psi_1^{(j)} = \left[-a\sin^2\psi f_{,rr} + \frac{\sin 2\psi}{a} f_{,\psi}\right]^{(j)},$$

$$\text{(b)}\quad {}_n\Psi_2^{(j)} = \left[\frac{a^2}{2}\sin^4\psi f_{,rrr} + \frac{a}{2}(2\sin^2\psi - \cos^2\psi)\sin^2\psi f_{,rr}\right.$$
$$\left. - \frac{1}{2}(\sin 2\psi)^2 f_{,r} - \sin^2\psi\sin 2\psi f_{,r\psi} + \frac{\sin 4\psi}{4a} f_{,\psi}\right]^{(j)}, \tag{A.5}$$

$$\text{(a)}\quad {}_s\Psi_1^{(j)} = \frac{1}{a}[\sin^2\psi(f_{,\psi} - af_{,\psi r}) - a\sin 2\psi f_{,r}]^{(j)},$$

$$\text{(b)}\quad {}_s\Psi_2^{(j)} = \frac{1}{a}\left[\frac{\sin^2\psi}{2}(a^2\sin^2\psi f_{,\psi rr} - \cos^2\psi\{3f_{,\psi} + af_{,\psi r}\} + 2a^2\sin 2\psi f_{,rr})\right.$$
$$\left. - \frac{a\sin 2\psi}{2}(\cos^2\psi - 3\sin^2\psi) f_{,r}\right]^{(j)}. \tag{A.6}$$